Algebra and Geometry of Whittaker patterns

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Lie Algebras and Invariant Theory

18 August 2018 , Samara

Motivation: compact/finite groups

The Peter-Weyl Theorem:

$$L^2(G) = \bigoplus_{\lambda \in \mathrm{Irr}(G)} V_\lambda \otimes V_\lambda^*\,,$$

summed over irreducible (finite-dimensional) representations $(\pi_\lambda,\, V_\lambda)$.

Here $V_\lambda \otimes V_\lambda^* =$ space of matrix elements $\langle e_i^\lambda,\, \pi_\lambda(g) e_j^\lambda \rangle_{V_\lambda}$, where

$$V_\lambda = \mathrm{span}\big\{e_1^\lambda, \ldots,\, e_{d(\lambda)}^\lambda\big\}\,, \qquad d(\lambda) = \dim(V_\lambda)$$

$$\langle\quad,\quad\rangle_{V_\lambda}\;:\quad V_\lambda \otimes V_\lambda^* \longrightarrow \mathbb{C}\,.$$

Matrix elements are well-known to be the source of
orthogonal polynomials/special functions

Principal series representations of reductive groups

The Bruhat (Gauss) decomposition of $G = G(F)$:

$$G \;=\; \bigsqcup_{w\in W} B_-\dot{w}B_-\,, \qquad G^0 \;=\; U_-\cdot A\cdot U_+\,.$$

For $\underline{\lambda} = (\lambda_1,\ldots,\lambda_N) \in \mathbb{C}^N$ consider general character of $B_- \subset G$,

$$\chi_{\underline{\lambda}}:\quad B_- = U_-A \longrightarrow \mathbb{C}^*\,, \qquad \chi_{\underline{\lambda}}(ua) \;=\; \prod_{i=1}^N |a_i|^{\lambda_i+\rho_i}\,.$$

The **principal series representation** $(\pi_{\underline{\lambda}},\,\mathcal{V}_{\underline{\lambda}})$ with the right G-action:

$$\mathcal{V}_{\underline{\lambda}} \;=\; \mathrm{Ind}_{B_-}^G\,\chi_{\underline{\lambda}} = \Big\{\,f\in L^2(G)\,\Big|\,f(bg) = \chi_{\underline{\lambda}}(b)\,f(g)\,,\quad b\in B_-\Big\}$$

There is an invariant non-degenerate pairing,

$$\langle\quad,\quad\rangle_{\mathcal{V}_{\underline{\lambda}}}:\quad \mathcal{V}_{\underline{\lambda}}\times\mathcal{V}_{\underline{\lambda}} \longrightarrow \mathbb{C}\,, \qquad \langle f,h\rangle_{\mathcal{V}_{\underline{\lambda}}} \;=\; \int_{U_+} d\mu_{U_+}(u)\,\overline{f(u)}\,h(u)\,.$$

The G-Whittaker function: definition

The general character $\psi_R:\ U_+ \longrightarrow \mathbb{C}^*$ is given by

$$\psi_R(u) \,=\, \prod_{\text{simple roots}} \psi\big(u_{\alpha_i}\big)\,, \qquad \psi\ :\quad F \longrightarrow \mathbb{C}^*\,.$$

The G-**Whittaker function** $\Psi_{\underline{\lambda}}(g)$ is a smooth function on $G(F)$ given by

$$\Psi_{\underline{\lambda}}(g) \,=\, \big\langle \psi_L\,,\, \pi_{\underline{\lambda}}(g)\,\psi_R \big\rangle_{\mathcal{V}_{\underline{\lambda}}}\,, \tag{1}$$

where the "left" Whittaker vector,

$$\psi_L\,:\,U_- \longrightarrow \mathbb{C}^*\,, \qquad \psi_L(u) \,=\, \psi_R\big(u\dot{w}_0^{-1}\big)^{-1}$$

is defined via the inner automorphism:

$$\iota\ :\quad U_+ \longrightarrow U_-\,, \qquad u \longmapsto \dot{w}_0\, u\, \dot{w}_0^{-1}\,.$$

Note: $\psi_L,\, \psi_R \,\in\, \mathcal{V}_{\underline{\lambda}}\,$!

The G-Whittaker function: basic properties

1. For any $u_1 \in U_-$ and $u_2 \in U_+$

$$\Psi_{\underline{\lambda}}(u_1 g u_2) \;=\; \psi_R(u_2)\,\Psi_{\underline{\lambda}}(g)\,;$$

2. For any G-invariant differential operator $\mathcal{H}$ on $A \subset G$

$$\mathcal{H}\cdot\Psi_{\underline{\lambda}}(a) \;=\; c_{\mathcal{H}}(\underline{\lambda})\,\Psi_{\underline{\lambda}}(a)\,,\qquad a\in A\,,\quad c_{\mathcal{H}}(\underline{\lambda})\in\mathbb{C}\,.$$

3. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra, then

$$\mathcal{V}_{\underline{\lambda}}\times\mathcal{U}(\mathfrak{g})\longrightarrow\mathcal{V}_{\underline{\lambda}}\,,\qquad \bigl(X\cdot f\bigr)(g) \;=\; \frac{d}{dt}f\left(ge^{tX}\right)\Bigr|_{t\to 0}\,,\quad \forall X\in\mathfrak{g}\,;$$

moreover, the $\mathcal{U}(\mathfrak{g})$-action is Hermitian w.r.t. the pairing:

$$\langle X\cdot f,\,h\rangle_{\mathcal{V}_{\underline{\lambda}}} \;=\; -\langle f,\,X\cdot h\rangle_{\mathcal{V}_{\underline{\lambda}}}\,;$$

4. Let $\mathrm{Lie}(U_+) = \mathrm{span}\{e_i,\, i\in I\}$ and $\mathrm{Lie}(U_-) = \mathrm{span}\{f_i,\, i\in I\}$, then

$$e_i\cdot\psi_R \;=\; \xi_i^+\psi_R\,,\qquad f_i\cdot\psi_L \;=\; \xi_i^-\psi_L\,,\qquad \xi_i^\pm\in\mathbb{C}\,.$$

The quantum Toda D-module

The G-Whittaker function on maximal torus $A \subset G$, so that $\dim(A) = \mathrm{rk}(G)$:

$$\Psi^{\mathfrak{g}}_{\lambda_1,\ldots,\lambda_\ell}(e^{x_1},\,\ldots,\,e^{x_\ell}) \;=\; e^{-\rho(x)}\big\langle\psi_L,\,\pi_{\underline{\lambda}}\big(e^{-\sum x_i h_i}\big)\cdot\psi_R\big\rangle_{\mathcal{V}_{\underline{\lambda}}}$$

Generators $C_1,\ldots,C_\ell$ of the center $\mathcal{Z}\mathcal{U}(\mathfrak{g})$ define quantum Toda Hamiltonians:

$$\mathcal{H}_r\cdot\Psi_{\underline{\lambda}}(e^x) \;:=\; e^{-\rho(x)}\big\langle\psi_K\,,\,\pi_{\underline{\lambda}}\big(C_r\,e^{-H(x)}\big)\,\psi_R\big\rangle\,. \tag{2}$$

The $G(\mathbb{R})$-Whittaker function is an eigenfunction:

$$\mathcal{H}_r\cdot\Psi_{\underline{\lambda}}(e^x) \;=\; e_r(\underline{\lambda})\,\Psi_{\underline{\lambda}}(e^x)\,, \tag{3}$$

$e_r(\underline{\lambda})$ are r-symmetric functions in $\underline{\lambda}=(\lambda_1,\ldots,\lambda_N)$.

Example: $G \;=\; GL(2;\,\mathbb{R})$

$$\mathcal{H}_1 \;=\; -\hbar\Big(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\Big)\,, \qquad\qquad \mathcal{H}_2 \;=\; -\hbar^2\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\;+\;e^{x_1-x_2}\,,$$

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Example: the $GL(2;\mathbb{R})$-Whittaker functions

The Bessel function "of the third kind":

$$\Psi^{\mathfrak{gl}_2}_{\lambda_1,\lambda_2}(e^{x_1},\, e^{x_2}) \,=\, \int_{\mathbb{R}} dT\; e^{\frac{\imath}{\hbar}\lambda_2(x_1+x_2-T)+\frac{\imath}{\hbar}\lambda_1 T-\frac{1}{\hbar}\left(e^{x_1-T}+e^{T-x_2}\right)} \tag{4}$$

$$=\, e^{\frac{\lambda_1+\lambda_2}{2}}\, e^{\frac{x_1+x_2}{2}}\; K_{\frac{\lambda_1-\lambda_2}{\hbar}}\Big(\frac{2}{\hbar}e^{\frac{x_1-x_2}{2}}\Big)\,.$$

The Mellin-Barnes integral representation:

$$\Psi^{\mathfrak{gl}_2}_{\lambda_1,\lambda_2}(e^{x_1},\, e^{x_2}) \,=\, \int\limits_{\mathbb{R}-\imath\epsilon} d\gamma\;\, e^{\frac{\imath}{\hbar}x_2(\lambda_1+\lambda_2-\gamma)+\frac{\imath}{\hbar}x_1\gamma}$$

$$\times\hbar^{\frac{\lambda_1-\gamma}{\hbar}}\,\Gamma\Big(\frac{\lambda_1-\gamma}{\hbar}\Big)\hbar^{\frac{\lambda_2-\gamma}{\hbar}}\,\Gamma\Big(\frac{\lambda_2-\gamma}{\hbar}\Big)\,. \tag{5}$$

Both integral representations can be generalized to $GL_N(\mathbb{R})$ by induction over the rank N, using the Baxter $\mathcal{Q}$-operator formalism, [GLO:08,09,14].

Algebraic construction of Whittaker vectors I

We have the following integral representation:

$$\Psi^{\mathfrak{g}}_{\underline{\lambda}}(e^x) \;=\; e^{-\rho(x)} \int_{U_+} d\mu_{U_+}(u)\, \overline{\psi_L(u)}\, \psi_R\big(ue^{-\sum x_i h_i}\big) \tag{6}$$

In case $G = GL_{\ell+1}(\mathbb{R})$ general unipotent character is given by

$$\begin{gathered}\psi_R \;:\quad U_+ \longrightarrow \mathbb{C}^*\,, \qquad \|u_{ij}\|_{i<j} \longmapsto e^{-\sum u_{i,i+1}}\,,\\ \psi_R(u_1)\psi_R(u_2) = \psi_R(u_1 u_2)\,.\end{gathered} \tag{7}$$

Chasing general case, compute $u_{i,i+1}$ in terms of $i \times i$-minors of $u \in U_+$

$$u_{i,i+1} \;=\; \frac{\Delta_i(u\dot{s}_i)}{\Delta_i(u)}\,, \qquad \psi_R(u) \;=\; e^{-\sum\limits_{i\in I} \frac{\Delta_i(u\dot{s}_i)}{\Delta_i(u)}}\;;$$

and using the conjugation by the longest Weyl group element $\dot{w}_0$

$$\psi_L(u) \;=\; \psi_R(u\dot{w}_0^{-1})^{-1} \;=\; \chi_{\underline{\lambda}}(u\dot{w}_0^{-1})\, e^{\sum\limits_{i\in I} \frac{\Delta_i(u\dot{w}_0^{-1}\dot{s}_i)}{\Delta_i(u\dot{w}_0^{-1})}}\,.$$

Algebraic construction of Whittaker vectors II

In general we have

$$\mathbb{C}[U_+] \simeq \bigoplus_{\lambda\in\Lambda_W^+} V_\lambda\,;$$

let $V_{\varpi_1}, \ldots, V_{\varpi_\ell}$ be fundamental representations,

$$\langle\ \ ,\ \ \rangle_{\varpi_i}\ :\quad V_{\varpi_i}\times V_{\varpi_i}\longrightarrow\mathbb{C}$$

such that for highest/lowest weight vectors $\xi^{\pm}_{\varpi_i}\in V_{\varpi_i}$

$$\langle\xi^+_{\varpi_i},\xi^+_{\varpi_i}\rangle_{\varpi_i}\,=\,1\,.$$

For each $(\pi_{\varpi_i},\, V_{\varpi_i})$ introduce

$$\Delta_{\varpi_i}(g)\,=\,\big\langle\xi^-_{\varpi_i},\,\pi_{\varpi_i}(g)\xi^+_{\varpi_i}\big\rangle_{\varpi_i}\,,$$

then it can be shown that

$$\psi_R(u)\,=\,\prod_{i\in I}e^{-u_{\alpha_i}}\,=\,e^{-\sum\limits_{i\in I}\frac{\Delta_{\varpi_i}(u\dot{s}_i)}{\Delta_{\varpi_i}(u)}}\,,$$

$$\psi_L(u)\,=\,\psi_R(u\dot{w}_0^{-1})^{-1}\,=\,\chi_{\underline{\lambda}}\big(u\dot{w}_0^{-1}\big)\,e^{\sum\limits_{i\in I}\frac{\Delta_{\varpi_i}(u\dot{w}_0^{-1}\dot{s}_i)}{\Delta_{\varpi_i}(u\dot{w}_0^{-1})}}\,.$$

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The Main Theorem ['96 '05 '07 '12]

Theorem

The G-Whittaker function allows for the following integral representation:

$$\Psi^{\mathfrak{g}}_{\underline{\lambda}}(e^x) \;=\; e^{-\rho(x)} \int_{U_+} d\mu_{U_+}(u) \prod_{i\in I} \Delta_{\varpi_i}\big(u\dot{w}_0^{-1}\big)^{\frac{\imath}{\hbar}\langle\underline{\lambda},\,\alpha_i^\vee\rangle} \times \exp\Big\{ \sum_{i\in I} \Big(\frac{\Delta_{\varpi_i}(u\dot{w}_0^{-1}\dot{s}_i)}{\Delta_{\varpi_i}(u\dot{w}_0^{-1})} \;-\; e^{\langle\alpha_i,\,x\rangle} \frac{\Delta_{\varpi_i}(u\dot{s}_i)}{\Delta_{\varpi_i}(u)} \Big) \Big\} \tag{8}$$

Example: $G \;=\; SL(2;\,\mathbb{R})$

$$u \;=\; \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \qquad \dot{w}_0 \;=\; \dot{s} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad u\dot{w}_0^{-1} \;=\; \begin{pmatrix} y & -1 \\ 1 & 0 \end{pmatrix},$$

so that (8) reads

$$\Psi^{\mathfrak{sl}_2}_{\lambda_1,\,\lambda_2}(e^{x_1},\, e^{x_2}) \;=\; e^{\frac{x_1-x_2}{2}} \int \frac{dy}{y}\, y^{\lambda_1-\lambda_2}\, e^{-\frac{1}{y} \,-\, e^{x_1-x_2}\, y}\,. \tag{9}$$

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The group épinage: elementary unipotent parameters

For each $i \in I$ introduce the group épinage:

$$\varphi_i : \quad SL_2 \longrightarrow G \,, \qquad X_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} .$$

Given a reduced word $J = (j_1, \ldots, j_m) \in R(w_0)$, so that $m = l(w_0) = |\Phi_+|$,

$$\Phi_+ = \big\{ \gamma_1 = \alpha_{j_1} \,, \quad \gamma_2 = s_{j_1} \cdot \alpha_{j_2} \,, \quad \ldots, \quad \gamma_m = (s_{j_1} \cdots s_{j_{m-1}) \cdot \alpha_{j_m}} \big\} \,;$$

moreover, there is a birational isomorphism

$$\mathbb{C}^m \longrightarrow U_+ \,, \qquad (t_1, \ldots, t_m) \longmapsto u = X_{j_1}(t_1) \cdot \ldots \cdot X_{j_m}(t_m) \,.$$

Then the generalised minors can be computed [Berenstein-Zelevinsky]:

$$\Delta_{\varpi_i}\big(u \dot{w}_0^{-1}\big) = \prod_{n=1}^{m} t_n^{\langle \varpi_i , \gamma_n^{\vee} \rangle} \tag{10}$$

Example: elementary unipotent parameters in type A_2, I

There are two reduced words of $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2 \in W(A_2) \simeq \mathfrak{S}_3$:

$$(1,2,1) \qquad \text{and} \qquad (2,1,2)$$

- In case $J = (1,2,1)$

$$u = \begin{pmatrix} 1 & t_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_1+t_3 & t_1 t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \dot{w}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and, therefore,

$$\frac{\Delta_{\varpi_1}(u\dot{w}_0^{-1}\dot{s}_1)}{\Delta_{\varpi_1}(u\dot{w}_0^{-1})} = -\frac{t_1+t_3}{t_1 t_2}, \qquad \frac{\Delta_{\varpi_2}(u\dot{w}_0^{-1}\dot{s}_2)}{\Delta_{\varpi_2}(u\dot{w}_0^{-1})} = -\frac{t_2}{t_2 t_3},$$

which leads to

$$\Psi^{\mathfrak{sl}_3}_{\lambda_1,\lambda_2,\lambda_3}(e^{x_1}, e^{x_2}, e^{x_3}) = e^{x_1-x_3}\int \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\,(t_1 t_2)^{\lambda_1-\lambda_2}(t_2 t_3)^{\lambda_2-\lambda_3}$$
$$\times \exp\Big\{-\frac{1}{t_2} - \frac{1}{t_1}\frac{t_3}{t_2} - \frac{1}{t_3} - e^{x_1-x_2}(t_1+t_3) - e^{x_2-x_3}t_2\Big\}$$

Example: elementary unipotent parameters in type A_2, II

- In case $J = (2, 1, 2)$

$$u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & q_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & q_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q_2 & q_2 q_3 \\ 0 & 1 & q_1 + q_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \dot{w}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

one might observe

$$q_1 = \tfrac{t_2 t_3}{t_1 + t_3}, \qquad q_2 = t_1 + t_3, \qquad q_3 = \tfrac{t_1 t_2}{1_1 + t_3},$$

$$\frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{t_3} = \frac{dq_1}{q_1} \frac{dq_2}{q_2} \frac{dq_3}{q_3}.$$

Therefore,

$$\frac{\Delta_{\varpi_1}(u \dot{w}_0^{-1} \dot{s}_1)}{\Delta_{\varpi_1}(u \dot{w}_0^{-1})} = -\frac{q_2}{q_2 q_3}, \quad \frac{\Delta_{\varpi_2}(u \dot{w}_0^{-1} \dot{s}_2)}{\Delta_{\varpi_2}(u \dot{w}_0^{-1})} = -\frac{q_1 + q_3}{q_1 q_2},$$

which leads to

$$\Psi^{\mathfrak{sl}_3}_{\lambda_1, \lambda_2, \lambda_3}(e^{x_1}, e^{x_2}, e^{x_3}) = e^{x_1 - x_3} \int \frac{dq_1}{q_1} \frac{dq_2}{q_2} \frac{dq_3}{q_3} (q_2 q_3)^{\lambda_1 - \lambda_2} (q_1 q_2)^{\lambda_2 - \lambda_3}$$
$$\times \exp\Big\{ -\frac{1}{q_3} - \frac{1}{q_2} - \frac{1}{q_1} \frac{q_3}{q_2} - e^{x_1 - x_2} q_2 - e^{x_2 - x_3}(q_1 + q_3) \Big\}$$

Givental integral representation in type A_ℓ [GLO'07 '12]

Let $J = (1, 21, \ldots, \ell \ldots 21) \in R(w_0)$,

$$u \;=\; u^{(\ell)} \cdot X_\ell(y_{\ell,1}) \cdots X_1(y_{1,\ell})$$

then making the **totally positive** substitution

$$y_{i,n} \;=\; e^{x_{n+i,\,i+1} - x_{n+i-1,\,i}}\,, \qquad 1 \le n \le i \le \ell$$

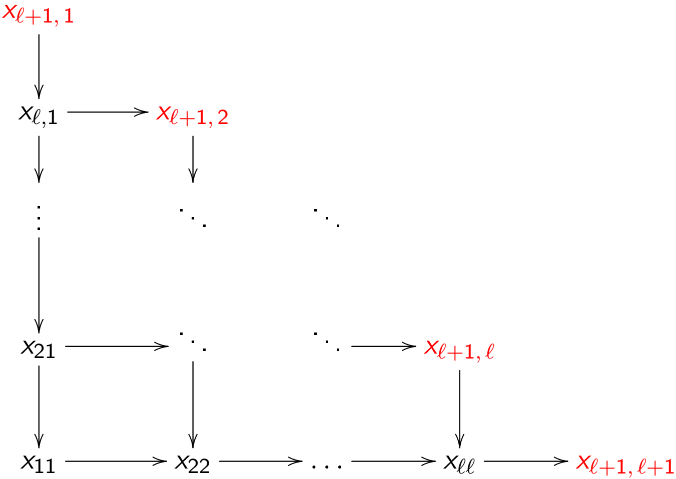
gives the following integral representation

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(e^{x_1}, \ldots, e^{x_{\ell+1}}) \;=\; \int_C \prod_{k=1}^{\ell} d\underline{x}_k\; e^{\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)}, \qquad (11)$$

where the function $\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)$ is given by

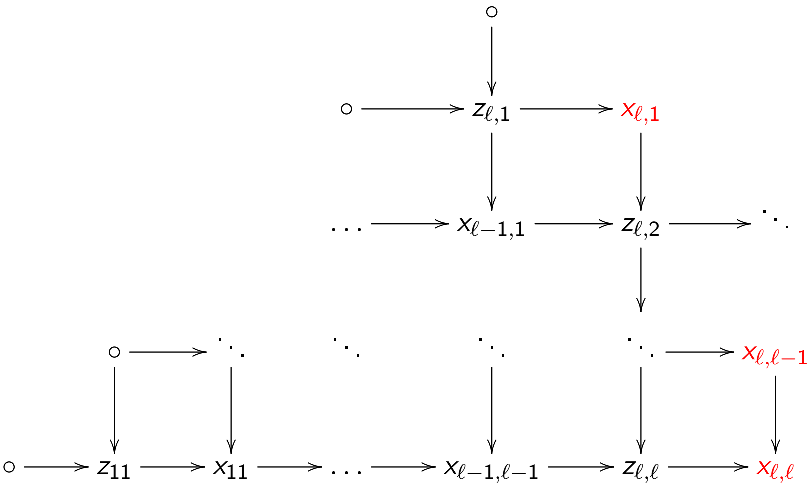
$$\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x) \;=\; \imath \sum_{n=1}^{\ell+1} \lambda_n \Big(\sum_{i=1}^{n} x_{n,i} - \sum_{i=1}^{n-1} x_{n-1,i} \Big)$$
$$- \sum_{k=1}^{\ell} \sum_{i=1}^{k} \Big(e^{x_{k,i} - x_{k+1,i}} + e^{x_{k+1,\,i+1} - x_{k,i}} \Big)\,.$$

Gelfand-Tsetlin graph in type A_ℓ [GL0'07 '12]



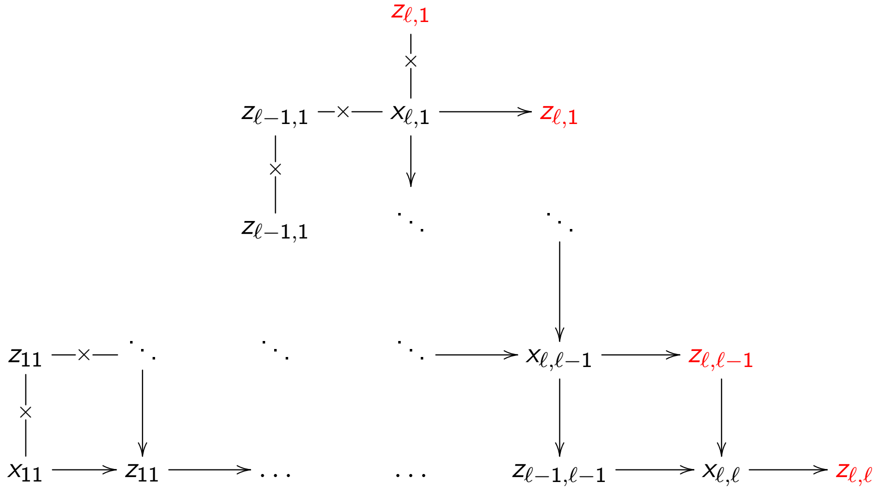
Gelfand-Tsetlin graph in type B_ℓ [GL0'07 '12]

Let $J = (j_1, j_2, \ldots, j_m) := (1, 212, 32123, \ldots, (\ell \ldots 212 \ldots \ell))$



Gelfand-Tsetlin graph in type C_ℓ [GLO'07 '12]

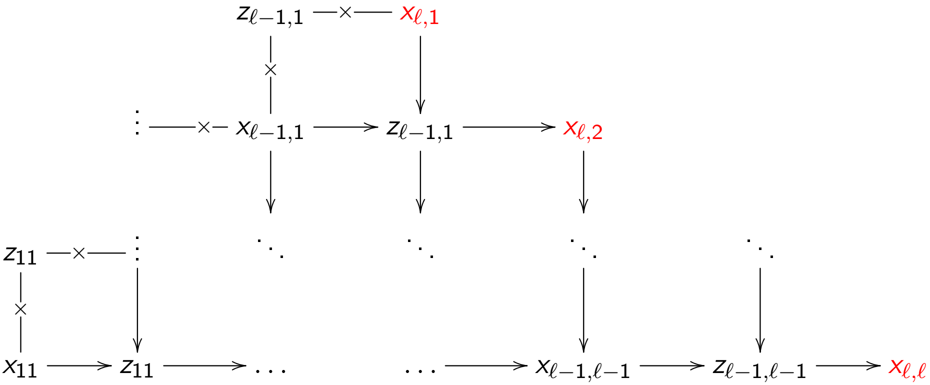
Let $J = (j_1, j_2, \ldots, j_m) := (1, 212, 32123, \ldots, (\ell \ldots 212 \ldots \ell))$



where we assign to the symbol $z \,—\!\times\!—\, x$ the exponent e^{-z-x}

Gelfand-Tsetlin graph in type D_ℓ [GL0'07 '12]

Let $J = (j_1, j_2, \dots, j_m) := (12, 3123, \dots, (\ell \dots 3123 \dots \ell))$



The C_ℓ-type VS D_ℓ-type symmetry :

The (twisted) affine Lie algebra of type $A^{(2)}_{2\ell-1}$: